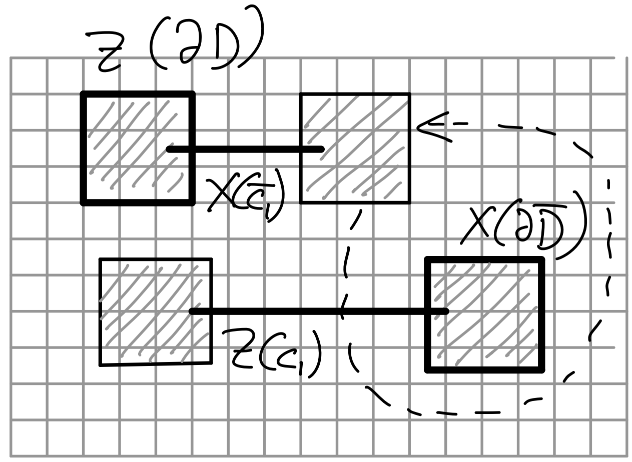
§4.3 Logical Controlled-NOT Gate by braiding

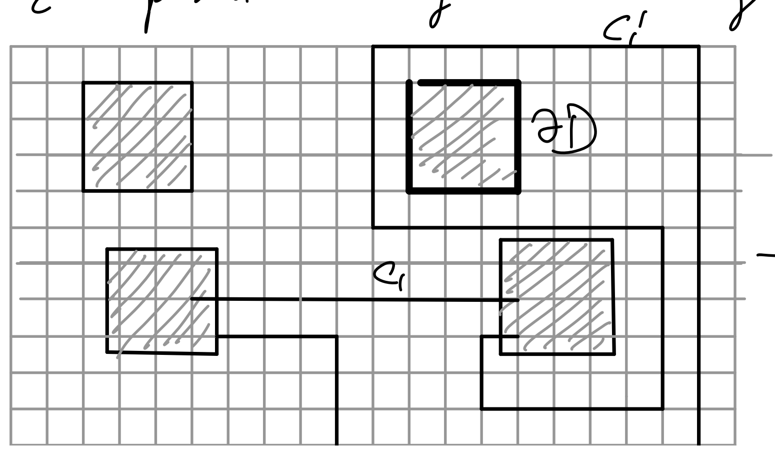
Consider CNOT gate between primal (control) and dual (target) defect pair qubits



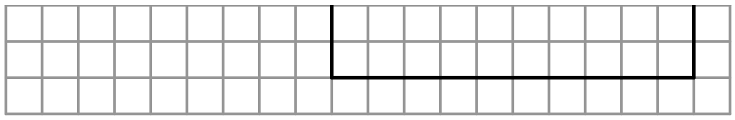
This operation transforms the $X(\overline{c_1})$ operator to

$$X(\overline{c_1}') \sim X(\overline{c_1})X(\partial\overline{D})$$

Similarly, one can induce a time evolution of the logical Z operator by braiding:



$\rightarrow Z(c_1') \sim Z(\partial D)Z(c_1)$



On the other hand $Z(\partial D)$ and $X(\partial \bar{D})$ are invariant under this operation.

To summarize, we get

$$
\begin{aligned}
Z(\partial D) &\longrightarrow Z(\partial D)\\
Z(c_1) &\longrightarrow Z(c_1)\, Z(\partial D)\\
X(\partial \bar{D}) &\longrightarrow X(\partial \bar{D})\\
X(\bar{c}_1) &\longrightarrow X(\bar{c}_1)\, X(\partial \bar{D})
\end{aligned}
$$

This is a CNOT-gate!

Recall:

$$\Lambda_{c,t}(X)\, X_c\, \Lambda_{c,t}(X) = X_c X_t$$

$$\Lambda_{c,t}(X)\, X_t\, \Lambda_{c,t}(X) = X_t$$

$$\Lambda_{c,t}(X)\, Z_c\, \Lambda_{c,t}(X) = Z_c$$

$$\Lambda_{c,t}(X)\, Z_t\, \Lambda_{c,t}(X) = Z_c Z_t$$

→ description limited as we are dealing with "Abelian" defects

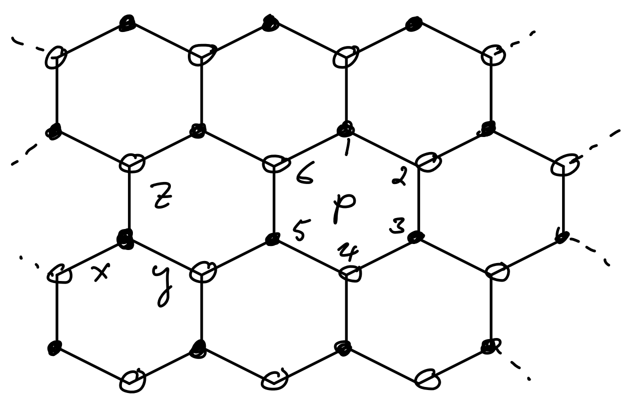
§5. Kitaev's Honeycomb lattice model

§5.1 Introducing the honeycomb lattice model

The spin lattice Hamiltonian

$$H = -J_x \sum_{x\text{ links}} \sigma_i^x \sigma_j^x - J_y \sum_{y\text{ links}} \sigma_i^y \sigma_j^y \qquad (*)$$

$$- J_z \sum_{z\text{ links}} \sigma_i^z \sigma_j^z - K \sum_{(i,j,k)} \sigma_i^x \sigma_j^y \sigma_k^z$$



effective magnetic field with coupling K

In the following we take the J couplings to be all equal: $J_x = J_y = J_z = J$

$\longrightarrow$ "non-Abelian" topological phase

K-term breaks time-reversal symmetry

$$\hat{T} \sigma_i^\alpha \hat{T}^\dagger = -\sigma_i^\alpha$$

anti-linear unitary operator

The Hamiltonian (*) has a local symmetry. Consider "plaquette operators"

$$\hat{w}_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$

$$\longrightarrow \quad \hat{w}_p^2 = \mathbb{1}$$

$\longrightarrow$ eigenvalues are $w_p = \pm 1$.

Moreover: $[\hat{w}_p, \hat{w}_{p'}] = 0 \quad \forall\, p, p'$

$$[H, \hat{w}_p] = 0 \quad \forall\, p$$

Since the $\hat{w}_p$ are conserved quantities, the Hilbert space $\mathcal{L}$ of N spins on a plane $\longrightarrow$ $2^{N/2}$ sectors $\mathcal{L}_w$ of dimension $2^{N/2}$ labeled by $w = \{w_p\}$

Majorana fermionization

Goal: bring Hamiltonian (*) to quadratic form

Introduce two fermionic modes for each spin $\frac{1}{2}$ particle: $a_{1,i}$ and $a_{2,i}$

→ decompose into "real" and "imaginary" parts:

$$c_i = a_{1,i} + a_{1,i}^\dagger \,, \quad b_i^x = i(a_{1,i}^\dagger - a_{1,i}), \quad b_i^y = a_{2,i} + a_{2,i}^\dagger$$

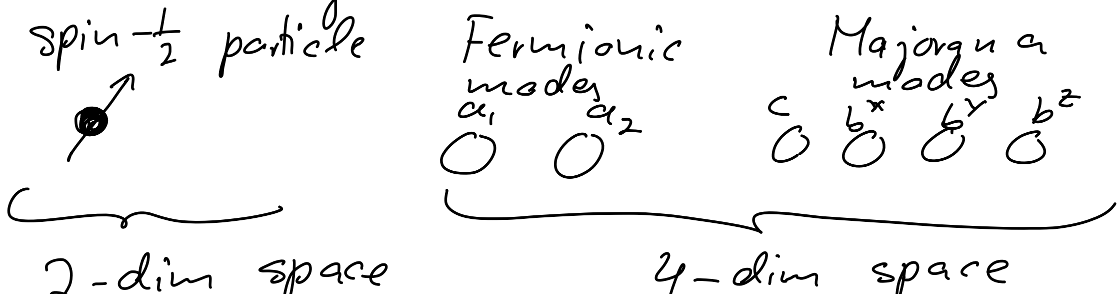
$$b_i^z = i(a_{2,i}^\dagger - a_{2,i})$$

Operators c_i, b_i^x, b_i^y, b_i^z are anti commuting and fermionic, satisfy "reality condition":

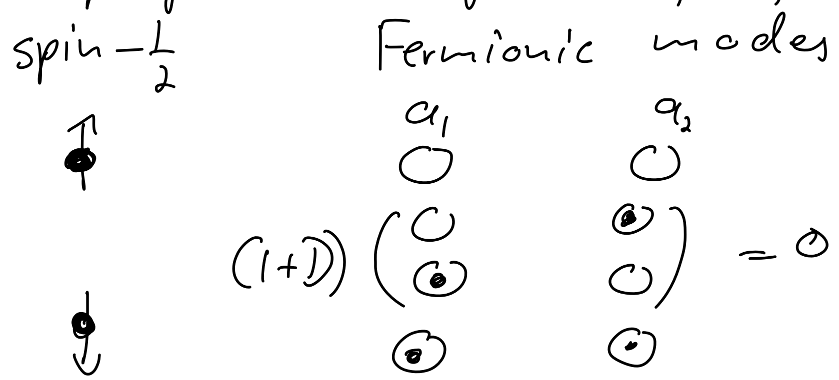
$$b_i^{\alpha\dagger} = b_i^\alpha \,, \quad c_i^\dagger = c_i$$

"Majorana fermions"

→ Dictionary:



→ need to project out 2 "unphysical" degrees of freedom



meaning:

$$|\uparrow\rangle = |00\rangle, \quad |\downarrow\rangle = |11\rangle$$

with $a_1 |00\rangle = a_2 |00\rangle = 0$, $|11\rangle = a_1^\dagger a_2^\dagger |00\rangle$

→ faithful if we restrict to subspace of ferm. states $|\psi\rangle$ satisfying!

$$D_i |\psi\rangle = |\psi\rangle$$

where $D_i = (1 - 2a_{1,i}^\dagger a_{1,i})(1 - 2a_{2,i}^\dagger a_{2,i}) = b_i^x b_i^y b_i^z c_i$

Here we can make following identification:

$$\sigma_i^\alpha = i b_i^\alpha c_i \quad \forall\ \alpha = x, y, z \qquad (**)$$

$$[D_i, \sigma_j^\alpha] = 0$$

and $i\sigma_i^x \sigma_i^y \sigma_i^z = b_i^x b_i^y b_i^z c_i = \mathbb{1}$.

→ satisfies the algebra of Pauli matrices only if we restrict to states that belong to $\mathcal{L}$.

Using the representation $(**)$, the Hamiltonian interactions become

$$\sigma_i^\alpha \sigma_j^\alpha = -i \hat{u}_{ij} c_i c_j \quad \text{and} \quad \sigma_i^x \sigma_j^y \sigma_k^z = -i \hat{u}_{ik} \hat{u}_{jk} D_k c_i c_j$$

where $\hat{u}_{ij} = i b_i^\alpha b_j^\alpha$ "link operators"

with $\alpha = x, y, z$

$\longrightarrow$ antisymmetric Hermitian operators satisfying

$$\hat{u}_{ij} = -\hat{u}_{ji}, \quad \hat{u}_{ij}^2 = 1, \quad \hat{u}_{ij}^\dagger = \hat{u}_{ij}$$

Restricting the states of the system to the physical space $\mathcal{L}$, the Hamiltonian (*) takes the form

$$H = \frac{i}{4} \sum_{i,j} \hat{A}_{ij} c_i c_j ,$$

$$\hat{A}_{ij} = 2J_{ij} \hat{u}_{ij} + 2K \sum_k \hat{u}_{ik} \hat{u}_{jk}$$

<u>Emerging lattice gauge theory</u>

It can be verified that

$$[H, D_i] = 0$$

Physical states must satisfy

$$H|\psi\rangle = E|\psi\rangle , \quad D_i|\psi\rangle = |\psi\rangle$$

The link operators are local symmetries:

$$[H, \hat{u}_{ij}] = 0$$

<u>But</u> $\{\hat{u}_{ij}, D_i\} = 0$

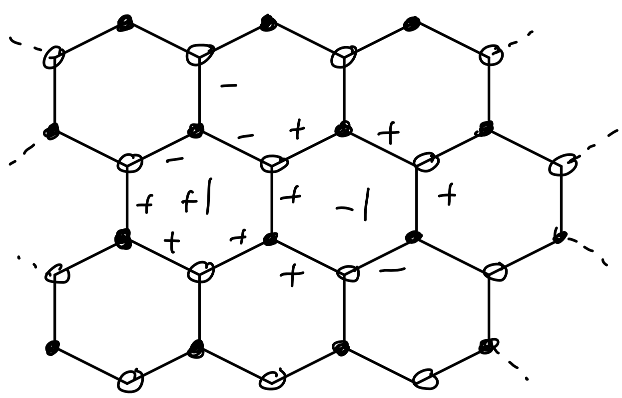
$\longrightarrow$ sectors labeled by eigenvalue patterns $u = \{u_{ij} = \pm\}$ are not part of $\mathcal{L}$

On the other hand, plaquette operators satisfy:

$$\hat{W}_p = \prod_{i,j \in p} \hat{u}_{ij}$$

$$[\hat{W}_p, H] = 0 \quad \text{and} \quad [\hat{W}_p, D_i] = 0$$

$\longrightarrow$ eigenvalues $\{u_{ij} = \pm 1\}$ can be thought of classical $\mathbb{Z}_2$ gauge field



$\longrightarrow$ pattern of vortices $(w_p = \pm 1)$ gauge invariant "Wilson loops"